Dynamic Asset Allocation with Predictable Returns and Transaction Costs

A  General quadratic objective function

It is straightforward to extend our approach to a non-zero risk-free rate $R_{0,t}$ and an objective function that is linear-quadratic in the position vector (i.e., $F(x_t, w_t) = w_T + a_1^T x_T - \frac{1}{2} x_T^T \Lambda T x_T$) rather than linear in total wealth. The $F(\cdot, \cdot)$ function parameters could be chosen to capture different objectives, such as maximizing the terminal gross value of the position ($w_T + 1^T x_T$) or the terminal liquidation (i.e., net of transaction costs) value of the portfolio ($w_T + 1^T x_T - \frac{1}{2} x_T^T \Lambda T x_T$), or the terminal wealth penalized for the riskiness of the position ($w_T + 1^T x_T - \frac{2}{3} x_T^T \Sigma T x_T$), or some intermediate situation.

Suppose the objective function is:

$$\max_{u_1, \ldots, u_T} \mathbb{E} \left[ F(w_T, x_T) - \sum_{t=0}^{T-1} \gamma^t x_t^T \Sigma_{t\rightarrow t+1} x_t \right]$$

By recursive substitution $x_T$ and $w_T$ can be rewritten as:

$$x_T = x_0 \circ R_{0\rightarrow T} + \sum_{t=1}^{T} u_t \circ R_{t\rightarrow T}$$

$$w_T = w_0 R_{0,0\rightarrow T} - \sum_{t=1}^{T} \left( u_t^T 1 R_{0,t\rightarrow T} + \frac{1}{2} u_t^T \Lambda_t u_t R_{0,t\rightarrow T} \right)$$

where we have defined security $i$’s cumulative return between date $t$ and $T$ as:

$$R_{i,t\rightarrow T} = \prod_{s=t+1}^{T} R_{i,s}$$

(with the convention that $R_{i,t\rightarrow t} = 1$) and the corresponding $N$-dimensional vector $R_{t\rightarrow T} = [R_{1,t\rightarrow T}; \ldots; R_{N,t\rightarrow T}]$.

Now note that:

$$a_1^T x_T = (a_1 \circ R_{0\rightarrow T})^T x_0 + \sum_{t=1}^{T} (a_1 \circ R_{t\rightarrow T})^T u_t$$
Substituting, we obtain the following:

\[
F(w_T, x_T) = F_0 + \sum_{t=1}^{T} \left\{ G_t^T u_t - \frac{1}{2} u_t^T P_t u_t \right\} - \frac{1}{2} x_T^T a_2 x_T \tag{56}
\]

\[
F_0 = w_0 R_{0,0 \rightarrow T} + (a_1 \circ R_{0 \rightarrow T})^T x_0 \tag{57}
\]

\[
G_t = a_1 \circ R_{t \rightarrow T} - 1 \circ R_{0,0 \rightarrow T} \tag{58}
\]

\[
P_t = \Lambda_t \circ R_{0,0 \rightarrow T} \tag{59}
\]

With these definitions, the objective function in equation (51) it can be rewritten as:

\[
F_0 - \frac{\gamma}{2} x_0^T Q_0 x_0 + \max_{u_1, \ldots, u_T} \sum_{t=1}^{T} \mathbb{E} \left[ G_t^T u_t - \frac{1}{2} u_t^T P_t u_t - \frac{\gamma}{2} x_t^T Q_t x_t \right] \tag{60}
\]

subject to the non-linear dynamics given in equations (4) and (5) and where we have defined

\[
Q_t = \begin{cases} 
    \Sigma_{t \rightarrow t+1} & \text{for } t < T \\
    \frac{1}{\gamma} a_2 & \text{for } t = T 
\end{cases} \tag{61}
\]

Indeed, substituting the definition of our linear trading strategies from equations (24) and (25) into our objective function in equation (60) and then taking expectations gives:

\[
F_0 - \frac{\gamma}{2} x_0^T Q_0 x_0 + \max_{\pi_1, \ldots, \pi_T} \sum_{t=1}^{T} G_t^T \pi_t - \frac{1}{2} \pi_t^T P_t \pi_t - \frac{\gamma}{2} \theta_t^T Q_t \theta_t \tag{62}
\]

subject to \( \theta_t = \theta_{t-1} + \pi_t \tag{63} \)

and where we define the vector \( G_t \) and the square matrices \( P_t \) and \( Q_t \) for \( t = 1, \ldots, T \) by

\[
G_t = \mathbb{E}_0[B_t G_t] \tag{64}
\]

\[
P_t = \mathbb{E}_0[B_t P_t B_t^T] \tag{65}
\]

\[
Q_t = \mathbb{E}_0[B_t Q_t B_t^T] \tag{66}
\]

Note that the time indices for \( G_t, P_t, Q_t \) also capture their size: \( G_t \) is a vector of length \( NK(t+1) \), and \( P_t \) and \( Q_t \) are square matrices of the same dimensionality.\(^{34}\) Equation (62) is just the objective function (equation (60)) with the \( u_t \)’s and \( x_t \)’s rewritten as linear functions.

\(^{34}\)It is important to note that these matrices \( G_t, P_t, Q_t \) will depend on the initial conditions (in particular on the initial exposures \( B_0 \), which typically will depend on the initial positions in each stock).
of the elements in $B_t$, with coefficients $\pi_t$ and $\theta_t$, respectively. Since the policy parameters $\pi_t$ and $\theta_t$ are set at time 0, they can be pulled outside of the expectation operator.

Intuitively equation (62) is a linear-quadratic function of the policy parameters $\pi_t$ and $\theta_t$, with $G_t$, $P_t$, $Q_t$ as the coefficients in this equation. These three components give, respectively, the effect on the objective function of: the expected portfolio returns resulting from trades at time $t$; the transaction costs paid as a result of trades at time $t$; and finally the effect of the holdings at time $t$ on the risk-penalty component of the objective function.

Since $G_t$, $P_t$, $Q_t$ are not functions of the policy parameters, they can be solved for explicitly or by simulation, and this only needs to be done once. Their values will depend on the initial conditions, and on the assumptions made about the state vector $X_t$ driving the return generating process $R_t$ and the corresponding security-specific exposure dynamics $B_{i,t}$. But, since equation (27) is a linear-quadratic equation, albeit a high-dimensional one, it can be solved using standard methods. We next calculate the closed form solution.

### A.1 Closed form solution

We begin with the linear-quadratic problem defined by equations (62) and (63). Define recursively the value function starting from $V(T) = 0$ for all $t \leq T$ by:

$$V(t - 1) = \max_{\pi_t} \left\{ G_t^T \pi_t - \frac{1}{2} \pi_t^T P_t \pi_t - \frac{\gamma}{2} \theta_t^T Q_t \theta_t + V(t) \right\}$$

subject to $\theta_t = \theta_{t-1}^0 + \pi_t$

Then it is clear that $V(0)$ is the solution to the problem we are seeking. To solve the problem explicitly, we guess that the value function is of the form:

$$V(t) = -\frac{\gamma}{2} \theta_t^T M_t \theta_t + L_t^T \theta_t + H_t$$

with $M_t$ a symmetric matrix. Since $V(T) = 0$, it follows that $M_T = 0$, $L_T = 0$ and $H_T = 0$. To find the recursion plug the guess in the Bellman equation:

$$V(t - 1) = \max_{\pi_t} \left\{ G_t^T \pi_t - \frac{1}{2} \pi_t^T P_t \pi_t - \frac{\gamma}{2} \theta_t^T (Q_t + M_t) \theta_t + L_t^T \theta_t + H_t \right\}$$

subject to $\theta_t = \theta_{t-1}^0 + \pi_t$

Now plugging in the constraint, we can simplify the Bellman equation using the following notation $\bar{x}$ is the vector (submatrix) obtained from $x$ by deleting the last $NK$ rows (rows
and columns). In Matlab notation \( x = x[1 : end - NK, 1 : end - NK] \).

\[
V(t - 1) = \max_{\pi_t} \left\{ (G_t + L_t)^\top \pi_t - \frac{1}{2} \pi_t^\top [P_t + \gamma(Q_t + M_t)] \pi_t - \frac{\gamma}{2} \theta_{t-1}^\top (Q_t + M_t) \theta_{t-1}
- \gamma \theta_{t-1}^\top (Q_t + M_t) \pi_t + \ell_i^\top \theta_{t-1} + H_t \right\}
\]

(68)

The first order condition gives:

\[
\pi_t = [P_t + \gamma(Q_t + M_t)]^{-1} (G_t + L_t - \gamma(Q_t + M_t)^\top \theta_{t-1}^0),
\]

and plugging into the state equation (equation (63)) we find

\[
\theta_t = [P_t + \gamma(Q_t + M_t)]^{-1} (G_t + L_t + P_t^\top \theta_{t-1}^0).
\]

Next, substitute these optimal policies into the Bellman equation in (68), giving:

\[
V(t - 1) = \frac{1}{2} (G_t + L_t - \gamma(Q_t + M_t)^\top \theta_{t-1}^0)^\top [P_t + \gamma(Q_t + M_t)]^{-1} (G_t + L_t - \gamma(Q_t + M_t)^\top \theta_{t-1}^0)
- \frac{\gamma}{2} \theta_{t-1}^\top (Q_t + M_t) \theta_{t-1} + H_t
\]

Setting \( \Psi_t = [P_t + \gamma(Q_t + M_t)]^{-1} \) and expanding we find:

\[
V(t - 1) = H_t + \frac{1}{2} (G_t + L_t)^\top \Psi_t (G_t + L_t)
- \gamma (G_t + L_t)^\top \Psi_t (Q_t + M_t)^\top \theta_{t-1}^0 + \ell_i^\top \theta_{t-1}
- \frac{\gamma}{2} \theta_{t-1}^\top \Psi_t (Q_t + M_t)^\top \Psi_t (Q_t + M_t) \theta_{t-1}
\]

Comparing this equation and the conjectured specification for \( V(t) \) in equation (67) shows that this specification will be correct if \( H_t, L_t, \) and \( M_t \) are chosen to satisfy the recursions:

\[
H_{t-1} = H_t + \frac{1}{2} (G_t + L_t)^\top \Psi_t (G_t + L_t)
\]

\[
L_{t-1} = L_t - \gamma (Q_t + M_t)^\top \Psi_t (G_t + L_t)
\]

\[
M_{t-1} = \ell_i^\top \theta_{t-1} - \gamma (Q_t + M_t)^\top \Psi_t (Q_t + M_t)
\]

with initial conditions \( H_T = 0, L_T = 0 \) and \( M_T = 0 \).

We have thus derived the optimal value function and the optimal trading strategy in the LGS class.
Before discussing some specific examples it is useful to introduce a set of LGS strategies which uses the exposures lagged at most \( \ell \) periods. This set of ‘restricted lag’ LGS is useful in applications when the time horizon is fairly long, and for signals that have a relatively fast decay rate, so that the dependence on lagged exposures can be restricted without a significant cost. We next show that the same tractability obtains for the restricted lag setting.

B  Constant variance of returns versus price changes

B.1  In dollars

Suppose \( x_t \) is vector of dollar holdings in risky shares and \( u_t \) is dollar trade at time \( t \). \( R_f \) is the risk-free rate and \( R_t \) is the vector of Gross returns. The net returns are given by \( r_t = R_t - 1 \) and \( r_f = R_f - 1 \).

Then we have with the convention that we trade at the end of the period:

\[
x_{t+1} = x_t \cdot R_{t+1} + u_{t+1} \quad (69)
\]
\[
W_{t+1} = W_t R_f + x_t' (R_{t+1} - R_f) - \frac{1}{2} u_{t+1} \Lambda_d u_{t+1} \quad (70)
\]

B.2  In shares

Suppose \( n_t \) is vector of number of shares held in risky shares and \( h_t \) is number of shares traded at time \( t \). \( R_f \) is the risk-free rate and \( dS_{t+1} = S_{t+1} - S_t \) is the vector of price changes (Assume no dividends for simplicity).

Then we have with the convention that we trade at the end of the period:

\[
n_{t+1} = n_t + h_{t+1} \quad (71)
\]
\[
W_{t+1} = W_t R_f + n_t' (dS_{t+1} - r_f S_t) - \frac{1}{2} h_{t+1} \Lambda_s h_{t+1} \quad (72)
\]

B.3  The objective function

For simplicity we set \( r_f = 0 \) and as in GP we solve the infinite horizon problem where the investor maximizes the discounted value of mean-variance objective functions.

In dollars

\[
E \left[ \sum_{t=1}^{\infty} \rho^t \left\{ x_t \mu_d - \frac{1}{2} u_t \Lambda_d u_t - \frac{\gamma}{2} x_t' \Sigma_d x_t \right\} \right] \quad (73)
\]

or, equivalently, in shares:
\[
\mathbb{E} \left[ \sum_{t=1}^{\infty} \rho^t \left\{ n_t \mu_s - \frac{1}{2} h_t \Lambda h_t - \frac{\gamma}{2} n_t' \Sigma_n n_t \right\} \right]
\]

Now, note that by definition:

\begin{align*}
x_t &= n_t \cdot S_t \quad (74) \\
u_t &= h_t \cdot S_t \quad (75) \\
\mu_s &= \mu_d \cdot S_t \quad (76) \\
\Sigma_s &= I_s \Sigma_d I_s \quad (77) \\
\Lambda_s &= I_s \Lambda_d I_s \quad (78)
\end{align*}

So clearly, assuming that the expectation and variance of dollar returns are constant is inconsistent with assuming that the expectation and variance of price changes are constant. We compare both cases next.

**B.4 Constant expectation and variance of dollar returns**

Let’s assume that the expectation and variance of returns are constant. Then it is helpful to introduce the state variable \( \bar{x}_t = x_t - u_t \), so that

\[
\bar{x}_{t+1} = (\bar{x}_t + u_t) \cdot R_{t+1}
\]

We can define the value function recursively by:

\[
J(\bar{x}_t) = \max_{u_t} \left\{ (\bar{x}_t + u_t) \mu_d - \frac{1}{2} u_t \Lambda_d u_t - \frac{\gamma}{2} (\bar{x}_t + u_t)' \Sigma_d (\bar{x}_t + u_t) + \rho \mathbb{E}_t [J(\bar{x}_{t+1})] \right\}
\]

Guess that the value function is quadratic.

\[
J(\bar{x}) = M_0 + M_1 \bar{x} + \bar{x}' M_2 \bar{x}
\]

Let’s first consider the one risky asset case. Then the solution is simply:

\[
u_t + \bar{x}_t = \frac{\bar{x}_t \Lambda_d + \mu_d + M_1 \rho \mu_d}{\Lambda_d + \gamma \Sigma_d - 2 \rho M_2 (\mu_d^2 + \Sigma_d)} =: a_0 + a_1 \bar{x}_t
\]

where the coefficient of the optimal value function are given by:
\[ M_2 = -\sqrt{\left(\gamma \Sigma - \Lambda (\rho (\mu^2 + \Sigma) - 1)\right)^2 + 4\gamma \Lambda \rho \Sigma (\mu^2 + \Sigma) - \gamma \Sigma + \Lambda (\rho (\mu^2 + \Sigma) - 1))} \frac{4\rho (\mu^2 + \Sigma)}{4\rho \mu} \] (82)

\[ M_1 = \sqrt{\left(\gamma \Sigma - \Lambda (\rho (\mu^2 + \Sigma) - 1)\right)^2 + 4\gamma \Lambda \rho \Sigma (\mu^2 + \Sigma) + \gamma \Sigma + \Lambda \mu^2 \rho - 2\Lambda \mu \rho + \Lambda \rho \Sigma + \Lambda} \] (83)

and \( M_0 \) can be computed explicitly, but is a rather lengthy expression.\(^{35}\) Note that

\[ a_1 = \frac{2\Lambda_d}{\Lambda (1 + \rho (\mu^2 + \Sigma)) + \gamma \Sigma_d + \sqrt{\left(\gamma \Sigma - \Lambda (\rho (\mu^2 + \Sigma) - 1)\right)^2 + 4\gamma \Lambda \rho \Sigma (\mu^2 + \Sigma)}} \]

Simple algebra confirms that \( a_1 \in (0, 1) \) if \( \gamma \Lambda \Sigma \rho > 0 \).

\section*{B.5 Constant expectation and variance of price changes}

For comparison purposes we make the same change of variables \( \pi_t = n_t - h_t \) so that

\[ \pi_{t+1} = \pi_t + h_t \]

Then we define the value function recursively by:

\[ J(\pi_t) = \max_{h_t} \left\{ (\pi_t + h_t)\mu_s - \frac{1}{2} h_t \Lambda_s h_t - \frac{\gamma}{2} (\pi_t + h_t)\Sigma_s (\pi_t + h_t) + \rho \mathbb{E}_t [J(\pi_{t+1})] \right\} \] (84)

Guess that the value function is quadratic.

\[ J(x) = N_0 + N'_t \pi + N_2 \pi \]

Let’s first consider the one risky asset case. Then we can solve everything in closed-form and we obtain:

\[ h_t + \pi_t = \frac{\pi_t \Lambda_s + \mu_s + N_1 \rho}{\Lambda_s + \gamma \Sigma_s - 2N_2 \rho} \] (85)

where the coefficient of the optimal value function are given by:

\(^{35}\)All calculations were made in Mathematica and the file is available upon request.
\[
N_2 = \frac{-\sqrt{(\gamma \Sigma + \Lambda(-\rho) + \Lambda)^2 + 4\gamma \Lambda \rho \Sigma + \gamma \Sigma + \Lambda(-\rho) + \Lambda}}{4 \rho}
\]
\[
N_1 = \frac{2 \Lambda \mu}{\sqrt{(\gamma \Sigma + \Lambda(-\rho) + \Lambda)^2 + 4\gamma \Lambda \rho \Sigma + \gamma \Sigma + \Lambda(-\rho) + \Lambda}}
\]

and

\[
N_0 = \left\{ -\frac{\mu^2 \left( (\rho - 1) \sqrt{\gamma^2 \Sigma^2 + 2\gamma \Lambda (\rho + 1) \Sigma + \Lambda^2 (\rho - 1)^2} + \gamma (\rho + 1) \Sigma + \Lambda (\rho - 1)^2 \right)}{4 \gamma^2 (\rho - 1) \rho \Sigma^2} \right\}
\]

\[\text{B.6 Comparing the two solutions}\]

The most obvious difference between the two solutions is that in the "constant expectation and variance of price change" case there exists a no-trade solution.

Indeed, solving for the fixed point \(\overline{n}_t\):

\[
\frac{\overline{n}_t \Lambda_s + \mu_s + N_1 \rho}{\Lambda_s + \gamma \Sigma_s - 2 N_2 \rho} = \overline{n}_t
\]

which is equivalent to

\[n_{no} = \frac{\mu}{\gamma \Sigma}\]

then we see that if \(\overline{n}_t = n_{no}\) at some time \(t\), then it is optimal to NEVER trade from then on, since \(h_t = 0\) and therefore \(\overline{n}_{t+s} = \overline{n}_{t+1} = \overline{n}_t = n_{no}\ \forall s > 0\) by induction. Instead, in the "constant expectation and variance of return" case, we see that the system can never settle into a no-trade equilibrium, since the dynamics of the state always lead to \(\overline{x}_{t+1} \neq \overline{x}_t\) even if \(u_t = 0\).

Further, it is interesting to note that the state where it is optimal not to trade for one period at time \(t\) in the "constant expectation and variance of return" case, is actually NOT the mean-variance efficient portfolio. Indeed, the no trade position for that case corresponds to a dollar position such that:

\[
\overline{x}_t = \frac{\overline{x}_t \Lambda_d + \mu_d + M_1 \rho \mu_d}{\Lambda_d + \gamma \Sigma_d - 2 M_2 \rho (\mu_d^2 + \Sigma_d)}
\]
Solving for \( x_{no} \) we find:

\[
x_{no} = \frac{2\mu (\mu^2 + \Sigma)}{(\mu_1 - \mu + \Sigma)\sqrt{\gamma^2 \Sigma^2 + 2 \gamma \Lambda \Sigma (\mu (\mu^2 + \Sigma) + 1) + \Lambda^2 (\mu (\mu^2 + \Sigma) - 1)^2 + \gamma \Lambda (\mu (\mu^2 + \Sigma) - 1)}}
\]

Note that \( x_{no} = \frac{\mu}{\gamma \Sigma} \) if \( \Lambda_d = 0 \) or if \( \rho = 0 \), but otherwise it is different!

Further, even if \( x_t = x_{no} \) at some \( t \) and thus \( u_t = 0 \) is optimal, since \( \pi_{t+1} = \pi_t R_{t+1} \) in that case, it will become optimal to trade at time \( t + 1 \).

### C Calibration of the Simulation Experiment

The RGPs for the characteristics and the factor environments (equations (40) and (41)) are, respectively

\[
R_{i,t+1} = \beta_{i,t}^T (F_{t+1} + \lambda) + \epsilon_{i,t+1}
\]

where \( E_t[F_{t+1}] = 0 \) and \( E_t[F_{t+1} F_{t+2}^T] = \Omega \) and

\[
R_{i,t+1} = \beta_{i,t}^T \lambda + \nu \epsilon_{i,t+1},
\]

where the factor exposures \( \beta_{i,t} \) and premia \( \lambda \) are each \((K, 1)\) vectors, and and where the evolution of the factor exposures is given by equation (40):

\[
\beta_{k,i,t+1}^k = (1 - \phi_k) \beta_{i,t}^k + \epsilon_{i,t+1},
\]

or equivalently:

\[
\beta_{i,t}^k = \sum_{s=0}^{\infty} (1 - \phi_k)^s \epsilon_{i,t-s}.
\]

Taken together, these imply, for either environment, that:

\[
E_t[R_{i,t+1}] = \beta_{i,t}^T \lambda
\]

\[
= \sum_{k=1}^{K} \lambda_k \beta_{i,t}^k
\]

\[
= \sum_{k=1}^{K} \lambda_k \sum_{s=0}^{\infty} (1 - \phi_k)^s \epsilon_{i,t-s}.
\]

In our simulation experiment in Section 3, we model the return-generating process for equities as consisting of \( K = 3 \) factors consistent with the short-term-reversal, medium-
term-momentum, and long-term-reversal effects. Consistent with the evidence on these three
effect, we choose half-lives for these factors of 5 days, 150 days, and 700 days.

To determine the parameters $\lambda$ and $\Omega$, we calibrate this factor model using the monthly
returns of portfolios formed on the basis of momentum, short- and long-term reversal, available on Ken French’s website. We use the full sample, 1927:01-2013:12. Note that data is
available on both the pre-formation and the post-formation returns of these sets of portfolios. We perform a Fama-MacBeth-like regression of the post-formation returns on the
pre-formation returns for each of the three sets of decile portfolios, and use the resulting
coefficients to estimate the set of $\lambda$s, given our assumed set of $\phi$s.

We characterize the slope coefficients for the three regressions with the formation period return horizons: our notation is that the formation period, for regression $j \in \{str, mom, ltr\}$, runs from time $t - m_j$ to $t - n_j$. For the characteristics model, the (cross-sectional) projection
of a one-day return onto a sum of returns from time $t - m_j$ to $t - n_j$ will give, under the
assumptions of our model:36

$$\text{cov} \left( R_{i,t+1}, \sum_{s=n_j}^{m_j} \epsilon_{i,t-s} \right) = \sigma_{\epsilon}^2 \sum_{k=1}^{3} \lambda_k \beta_{i,t}^k$$

$$= \sigma_{\epsilon}^2 \sum_{k=1}^{3} \sum_{s=n_j}^{m_j} \lambda_k (1 - \phi_k)^s$$

and

$$\text{var} \left( \sum_{s=n_j}^{m_j} \epsilon_{i,t-s} \right) = (m_j - n_j + 1)\sigma_{\epsilon}^2.$$

and finally

$$\beta_j = \frac{\text{cov} \left( R_{i,t+1}, \sum_{s=n_j}^{m_j} \epsilon_{i,t-s} \right)}{\text{var} \left( \sum_{s=n_j}^{m_j} \epsilon_{i,t-s} \right)} = \frac{3}{\sum_{k=1}^{3} \lambda_k} \frac{1}{(m_j - n_j + 1)} \sum_{s=n_j}^{m_j} (1 - \phi_k)^s.$$

$$= \sum_{k=1}^{3} \left( \frac{(1 - \phi_k)^{n_j} - (1 - \phi_k)^{m_j+1}}{\phi_k (m_j - n_j + 1)} \right) \lambda_k$$

$$= \sum_{k=1}^{3} a_{j,k} \lambda_k$$

36In practice we actually calculate the betas using returns rather than residuals. However, given that,
in the data particularly at short horizons, most of the variance of returns is idiosyncratic as opposed to
expected return variation, this approximation seems reasonable.
where

\[ a_{j,k} = \left( \frac{(1 - \phi_k)^{m_j} - (1 - \phi_k)^{m_j+1}}{\phi_k(m_j - n_j + 1)} \right) \]  

(91)

We find the three values of \( \lambda_k \) by solving the set of linear equations (for the three empirically estimated \( \beta_j \)s).

\[
\begin{bmatrix}
\beta_{str} \\
\beta_{mom} \\
\beta_{ltr}
\end{bmatrix} =
\begin{bmatrix}
a_{1,1} & a_{1,2} & a_{1,3} \\
a_{2,1} & a_{2,2} & a_{2,3} \\
a_{3,1} & a_{3,2} & a_{3,3}
\end{bmatrix}
\cdot
\begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3
\end{bmatrix}
\]

\( \lambda \) Estimation:

The Fama-MacBeth regressions yield (average) coefficients of:

\[
\begin{bmatrix}
\beta_{str} \\
\beta_{mom} \\
\beta_{ltr}
\end{bmatrix} =
\begin{bmatrix}
-0.00116273 \\
0.00044366 \\
-0.00010126
\end{bmatrix}
\]

The resulting \( \lambda \) estimates are:

\[
\begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3
\end{bmatrix} =
\begin{bmatrix}
-0.093482 \\
0.001484 \\
-0.000400
\end{bmatrix}
\]

\( \Omega \) Calibration:

The goal in the \( \Omega \) calibration is to come up with an upper bound on the magnitude of the covariance matrix. We employ the following procedure to estimate the \( 3 \times 3 \) factor covariance matrix \( \Omega \) using the three sets of decile portfolio returns: \( \text{str, mom, and ltr} \).

First, we use only the excess returns of the zero-investment portfolios formed by going long the top decile and short the bottom decile (\( i.e., \) the \( 10-1 \) portfolios). The factor loadings for these excess return portfolios are (from equation (40))

\[
\beta_{j,k,t}^{10-1} = \sum_{s=0}^{\infty} (1 - \phi_k)^s \epsilon_{j,t-s}^{10-1}
\]

Here, \( j \in \{ \text{str, mom, ltr} \} \) is French’s portfolio formation method; \( k \in \{1, 2, 3\} \) is the factor identifier, and \( t \) is the time (end-of-period) at which we are measuring the factor loading. As
in the preceding section, \( t - n_j \) and \( t - m_j \) are the starting and ending times for the period over which the pre-formation returns are measured for portfolio \( j \).

We are going to make several assumptions to allow the calculation of the factor loadings for each of these three portfolios. First, since portfolio \( j \) is formed on the basis of individual firm returns from \( t - m_j \) to \( t - n_j \), we assume that the residual returns for the portfolios are zero outside of that time range. This means that:

\[
\beta_{j,k,t}^{10-1} = \sum_{s=n_j}^{m_j} (1 - \phi_k)^s \epsilon_{j,t-s}^{10-1}
\]

Second, note that French only provides the formation period return on an annual basis. So, for example, for the LHR portfolios we have their cumulative return from \( t-60 \) months through \( t-12 \) months. So we assume that the average return was earned equally over each day in the 48 month period. If we denote the total pre-formation return as \( R_{\text{pre}}^{pre} \), we assume that the daily return, for each day in the 4 year period, was \( R_{\text{pre}}^{pre}/(4 \cdot 252) \). In general, given a \( 10-1 \) differential pre-formation return for strategy \( j \) in year \( y \) of \( R_{j,y}^{pre,10-1} \), we calculated the each daily return over the formation period as:

\[
R_{j,s}^{pre,10-1} = \frac{R_{j,y}^{pre,10-1}}{(m_j - n_j + 1)}
\]

for each day \( s \) between \( t - m_j \) and \( t - n_j \), and zero outside of the formation period. This means that the factor loading for portfolio \( 10-1 \) portfolio \( j \) on factor \( k \) is:

\[
\beta_{j,k,t}^{10-1} = \frac{R_{j,y}^{pre,10-1}}{(m_j - n_j + 1)} \sum_{s=n_j}^{m_j} (1 - \phi_k)^s \forall t \in y
\]

\[
= \left( \frac{(1 - \phi_k)^{m_j} - (1 - \phi_k)^{m_j+1}}{\phi_k(m_j - n_j + 1)} \right) R_{j,y}^{pre,10-1} \forall t \in y
\]

\[
= a_{j,k} R_{j,y}^{pre,10-1}
\]

where \( a_{j,k} \) is defined in equation (91).

Next, we assume that, since these are relatively well diversified portfolios, the residual risk \( (\sigma_r^2) \) is zero and further assume that all of the systematic risk comes from the three factors. These two assumptions imply that the covariance matrix for the time \( t + 1 \) returns
of the three $10-1$ portfolios, which we denote $\Sigma_t$, is given by:

$$\Sigma_t = \beta_t \Omega_t \beta_t^T$$

where

$$\beta_t = \begin{bmatrix} \beta_{str,1,t}^{10-1} & \beta_{str,2,t}^{10-1} & \beta_{str,3,t}^{10-1} \\ \beta_{mom,1,t}^{10-1} & \beta_{mom,2,t}^{10-1} & \beta_{mom,3,t}^{10-1} \\ \beta_{ltr,1,t}^{10-1} & \beta_{ltr,2,t}^{10-1} & \beta_{ltr,3,t}^{10-1} \end{bmatrix}$$

Note that this system is just identified, and $\Omega$ is given by:

$$\Omega = (\beta_t^\top \beta_t)^{-1} \beta_t^\top \Sigma_t \beta_t (\beta_t^\top \beta_t)^{-1}$$

We can estimate this either using the full sample covariance and the average pre-formation returns, or year-by-year and average the results.

Over the full-sample the average daily volatility of the daily $10-1$ portfolio returns are (in basis points):

$$\begin{bmatrix} \sigma_{str} \\ \sigma_{mom} \\ \sigma_{ltr} \end{bmatrix} = \begin{bmatrix} 28.464 \\ 37.817 \\ 30.367 \end{bmatrix}$$

and the correlation matrix of the returns is:

$$\begin{bmatrix} 1 & 0.250744 & 0.087098 \\ 0.250744 & 1 & 0.333539 \\ 0.087098 & 0.333539 & 1 \end{bmatrix}$$

The factor loading matrix for these three portfolios is:

$$B = \begin{bmatrix} 0.007291874 & 0.2927041 & 0.3146322 \\ 1.974574 \times 10^{-05} & 0.6481128 & 1.0529 \\ 1.061207 \times 10^{-28} & 0.2732635 & 2.100848 \end{bmatrix}$$

(92)

giving an estimated $\hat{\Omega}$ of:

$$\hat{\Omega} = \begin{bmatrix} 0.1655572 & -0.001041718 & 0.000119914 \\ -0.001041718 & 4.898553 \times 10^{-05} & -7.10805 \times 10^{-06} \\ 0.000119914 & -7.10805 \times 10^{-06} & 3.109768 \times 10^{-06} \end{bmatrix}$$
Or, decomposing this, the (daily) factor volatilities are:

\[
\hat{\sigma}_f = \begin{bmatrix}
0.4068872 \\
0.0069990 \\
0.0017635
\end{bmatrix}
\]

and the correlation matrix of the factors is estimated to be:

\[
\hat{\rho} = \begin{bmatrix}
1 & -0.3657987 & 0.1671214 \\
-0.3657987 & 1 & -0.5759073 \\
0.1671214 & -0.5759073 & 1
\end{bmatrix}
\]

\[^{37}\text{Note that the first factor has a large volatility (40%/day). This is a result of the way that we define the factor loadings in equation (40), where a firm’s factor loading is an exponentially weighted sum of past residual returns. When }\phi^k\text{ is large, as it is for }k = 1\text{, the dispersion in factor loadings across firms in the economy will be small. This is apparent in equation (92). Thus, a large factor volatility is required to explain the volatility of the long-short str volatility of only 28 bp/days.}\]